I - Basics of topology

PSL Week - Topological Data Analysis

Abstract

We introduce the basic topological ideas that underlie *topological data analysis* (TDA). For simplicity and conciseness, we restrict ourselves to the setting of *metric spaces*. We cover the notion of metric space, as well as two major notions of topological equivalence: homeomorphism, homotopy.

Contents

1 Metric spaces and basic examples

Definition 1.1 (Metric space)**.** A *metric space* is a pair (X, d) where X is a set and $d : X \times X \to [0, \infty)$ is a function (called a *distance* or *metric*) such that for all $x, y, z \in X$:

(i) $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$ (symmetry),

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

Example 1.2 (Euclidean space)**.** For $d \geq 1$, the space $\mathbb{R}^d$ with the distance

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_d - y_d)^2}$$

is a metric space. Most point clouds in applications live in (subsets of) Euclidean spaces.

Example 1.3 (Subspaces of a metric space)**.** Let (X, d) be a metric space and let $A \subset X$. Then $(A, d|_{A \times A})$ is also a metric space, called a *subspace* of X.

Example 1.4 (Discrete metric)**.** Let X be any set. The function

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y \end{cases}$$

defines a metric on X, called the *discrete metric*.

Example 1.5 (Graph distance)**.** Let $G = (V, E)$ be a (finite) graph without loops. Define $d(v, w)$ to be the length (number of edges) of a shortest path between v and w, and $d(v, w) = \infty$ if no such path exists. On each connected component of G this defines a metric; one can work componentwise.

Definition 1.6 (Open and closed balls)**.** Let (X, d) be a metric space, $x \in X$ and $r > 0$.

- The *open ball* of center x and radius r is

$$B_r(x) := \{y \in X : d(x, y) < r\}.$$

- The *closed ball* of center x and radius r is

$$\overline{B}_r(x) := \{y \in X : d(x, y) \leq r\}.$$

Definition 1.7 (Open and closed subsets)**.** Let (X, d) be a metric space.

- A subset $U \subset X$ is called *open* if for every $x \in U$ there exists $r > 0$ such that $B_r(x) \subset U$.
- A subset $F \subset X$ is called *closed* if its complement $X \setminus F$ is open.

Remark 1.8. In metric spaces one can show that a subset is closed if and only if it contains all its limit points, see below.

2 Convergence and continuity

Definition 2.1 (Convergence)**.** Let (X, d) be a metric space, and $(x_n)_{n\geq 1}$ a sequence in X. We say that x_n *converges* to $x \in X$, and write $x_n \to x$, if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n \geq N$.

Definition 2.2 (Closure)**.** Let $A \subset X$. A point $x \in X$ is called a *limit point* of A if there exists a sequence $(a_n)_n$ in A such that $a_n \to x$. The *closure* of A is

$$\overline{A} := \{x \in X : \exists\ (a_n)_n \subset A \text{ such that } a_n \to x\}.$$

Proposition 2.3. *A subset $F \subset X$ is closed if and only if $F = \overline{F}$.*

Idea. If F is closed and $(x_n)_n \subset F$ converges to x, then one can show $x \in F$. Conversely, if F contains all its limit points, its complement is open. The details are standard and similar to the Euclidean case. □

Definition 2.4 (Continuous map)**.** Let (X, d_X) and (Y, d_Y) be metric spaces, and $f : X \to Y$ a map. We say that f is *continuous at* $x_0 \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d_X(x, x_0) < \delta \quad \Rightarrow \quad d_Y\big(f(x), f(x_0)\big) < \varepsilon.$$

We say that f is *continuous* if it is continuous at every $x_0 \in X$.

Proposition 2.5 (Sequential characterization)**.** *A map $f : X \to Y$ between metric spaces is continuous if and only if for every sequence $(x_n)_n$ converging to x in X, the sequence $(f(x_n))_n$ converges to $f(x)$ in Y.*

Example 2.6 (Basic examples)**.**

(a) Polynomial maps $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - 2x + 1$, are continuous.

(b) The projection $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$, $\pi_1(x, y) = x$, is continuous.

(c) The map $f : \mathbb{R} \to \mathbb{R}^2$, $f(t) = (\cos 2\pi t, \sin 2\pi t)$ is continuous and parametrizes the unit circle S^1.

3 Homeomorphisms and topological properties

Informally, two spaces are *topologically the same* if they can be continuously deformed into each other without tearing or gluing. The precise notion is that of a homeomorphism.

Definition 3.1 (Homeomorphism)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \to Y$ is a *homeomorphism* if:

(i) f is bijective,

(ii) f is continuous,

(iii) the inverse map $f^{-1} : Y \to X$ is continuous.

If there exists a homeomorphism $f : X \to Y$, we say that X and Y are *homeomorphic* and write $X \simeq Y$.

Example 3.2 (Circle and square)**.** Consider the unit circle $S^1 \subset \mathbb{R}^2$ and the boundary of the unit square $C \subset \mathbb{R}^2$. Both are "simple closed curves" in the plane. One can construct a bijective map $f : S^1 \to C$ that is continuous with continuous inverse (for instance, via polar coordinates and projecting radially onto the square). Hence $S^1 \simeq C$.

Geometrically: you can continuously deform a circle into a square without cutting or gluing.

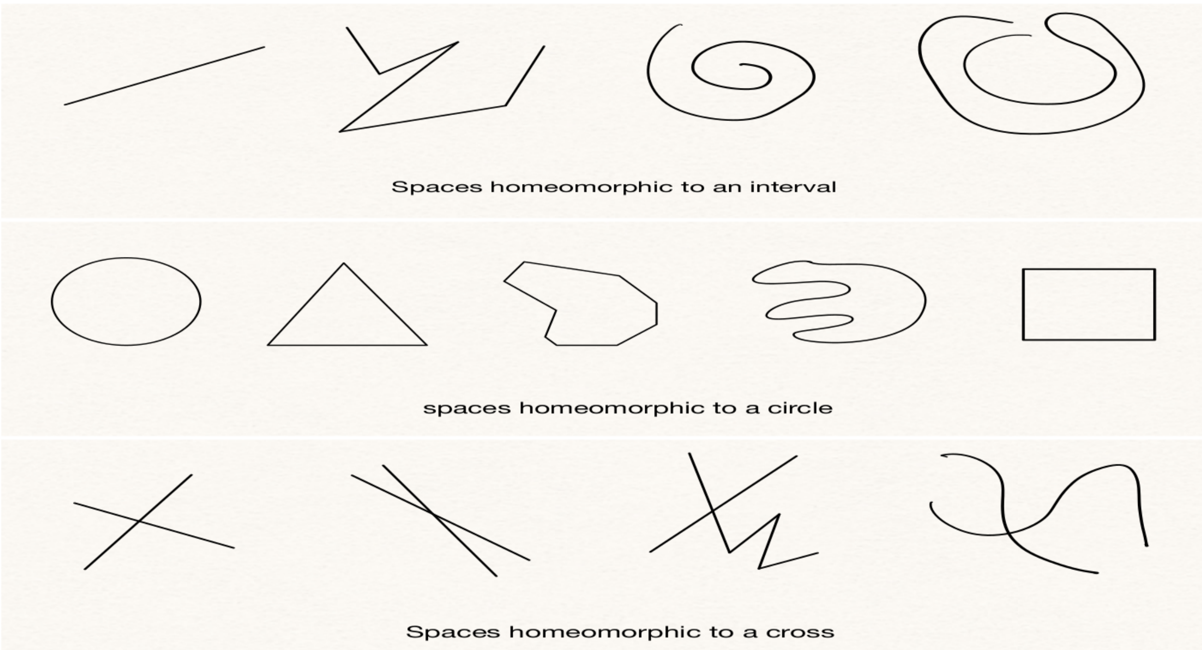

Figure 1: Three series of subsets of $\mathbb{R}^2$ homeomorphic to each other.

Example 3.3 (Interval and line)**.** The open interval $(0, 1)$ is homeomorphic to $\mathbb{R}$, via a bijective continuous map with continuous inverse, for example

$$\phi : (0, 1) \to \mathbb{R}, \qquad \phi(x) = \tan\big(\pi(x - \tfrac{1}{2})\big).$$

Intuitively, both spaces are "one-dimensional curves without endpoints".

Example 3.4 (Closed interval and circle are not homeomorphic)**.** The closed interval $[0, 1]$ is not homeomorphic to the circle S^1. One reason is that $[0, 1]$ has two "endpoints", whereas every point on S^1 looks the same. Formally, removing a single point from S^1 leaves a (connected) open arc. More formally, invariants such as the number of connected components of $X \setminus \{x\}$ (as x varies) differ.

Definition 3.5 (Topological property)**.** A property $\mathcal{P}$ of metric spaces is called *topological* if whenever (X, d_X) and (Y, d_Y) are homeomorphic, then X has $\mathcal{P}$ if and only if Y has $\mathcal{P}$.

Typical topological properties include: connectedness, path-connectedness, compactness, the number of connected components, the number and types of holes, etc. In contrast, metric notions such as "exact distances", angles, or volumes are *not* topological.

4 Connectedness and path-connectedness

Definition 4.1 (Connected metric space)**.** A metric space (X, d) is *disconnected* if there exist two non-empty disjoint open sets $U, V \subset X$ such that $X = U \cup V$. If no such decomposition exists, X is *connected.*

Definition 4.2 (Path)**.** Let (X, d) be a metric space. A *path* from x to y in X is a continuous map $\gamma : [0, 1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Definition 4.3 (Path-connected metric space)**.** A metric space (X, d) is *path-connected* if for every pair of points $x, y \in X$ there exists a path from x to y.

Proposition 4.4. *Every path-connected metric space is connected.*

Idea. Suppose X is path-connected and $X = U \cup V$ with U, V non-empty disjoint open sets. Let $x \in U$ and $y \in V$. Any path γ from x to y must at some point "cross" from U to V, which contradicts the openness of U and V. A formal proof uses the intermediate value property for the continuous function $t \mapsto \mathbf{1}_U(\gamma(t))$. $\square$

Example 4.5 (Basic examples)**.**

(a) Any interval in $\mathbb{R}$, open or closed, is path-connected (and hence connected).

(b) The union of two disjoint closed balls in $\mathbb{R}^2$ is not connected: there are two connected components.

(c) The unit circle S^1 is path-connected.

5 Homotopy equivalence between maps

To compare spaces at a coarser level than homeomorphism, and to describe continuous deformations, we introduce the notion of homotopy.

Definition 5.1 (Homotopy of maps)**.** Let X, Y be metric spaces and $f, g : X \to Y$ two continuous maps. A *homotopy* between f and g is a continuous map

$$H : X \times [0, 1] \to Y$$

such that

$$H(x, 0) = f(x) \quad \text{and} \quad H(x, 1) = g(x) \qquad \text{for all } x \in X.$$

We then say that f and g are *homotopic*, and write $f \simeq g$.

Remark 5.2. You can think of $t \in [0, 1]$ as "time": for each fixed t the map $H_t : X \to Y$, $H_t(x) := H(x, t)$ is a continuous map, and $(H_t)_{t\in[0,1]}$ gives a continuous deformation from f to g.

Example 5.3 (Straight-line homotopies in convex sets)**.** If X is a metric space and Y is convex, then all continuous functions $f, g : X \to Y$ are homotopy equivalent. Indeed, for all $t \in [0, 1]$, define

$$H(x, t) = (1 - t)f(x) + t\,g(x).$$

Then H is continuous and provides a homotopy from f to g.

Example 5.4 (Two non-homotopic functions)**.** If X is a metric space and $Y = [0, 1] \setminus \{1/2\}$, then the functions $f(x) = 1$ and $g(x) = 0$ are not homotopy equivalent to each other. Essentially, the reason is that any homotopy $H(t, x)$ would need to cross $1/2$, which does not belong to Y. This comes from the fact that Y is not path-connected.

6 Homotopy equivalence and deformation retracts

Homotopy allows us to define a coarser equivalence between spaces than homeomorphism, which is crucial in algebraic topology and TDA.

6.1 Deformation retracts

Definition 6.1 (Deformation retract)**.** Let X be a metric space and $A \subset X$ a subspace. We say that A is a *deformation retract* of X if there exists a continuous map

$$H : X \times [0, 1] \to X$$

such that:

(i) $H(x, 0) = x$ for all $x \in X$ (so $H_0 = \mathrm{id}_X$),

(ii) $H(a, t) = a$ for all $a \in A$ and all $t \in [0, 1]$ (points of A remain fixed during the deformation),

(iii) $H(x, 1) \in A$ for all $x \in X$, and $H(\,\cdot\,, 1)$ maps X *onto* A.

In this case we say that X *deformation retracts* onto A.

Example 6.2 (Disk retracting onto its centre)**.** Let $X = \overline{B}_1(0) \subset \mathbb{R}^d$ be the closed unit ball and $A = \{0\}$. Define

$$H(x, t) = (1 - t)x, \qquad x \in X,\ t \in [0, 1].$$

Then $H_0(x) = x$, $H_1(x) = 0 \in A$, and $H(0, t) = 0$ for all t. Hence $\{0\}$ is a deformation retract of $\overline{B}_1(0)$.

Example 6.3 (Cylinder retracting onto a circle)**.** Let $X = S^1 \times [0, 1] \subset \mathbb{R}^3$ be a cylinder (a circle extended in the vertical direction), and let $A = S^1 \times \{0\} \subset X$ be its bottom circle. Define

$$H\big((x, s), t\big) = (x, (1 - t)s), \qquad x \in S^1,\ s \in [0, 1].$$

Then $H_0 = \mathrm{id}_X$, $H_1((x, s)) = (x, 0) \in A$, and points of A stay fixed. Hence the cylinder deformation retracts onto the circle $S^1 \times \{0\}$.

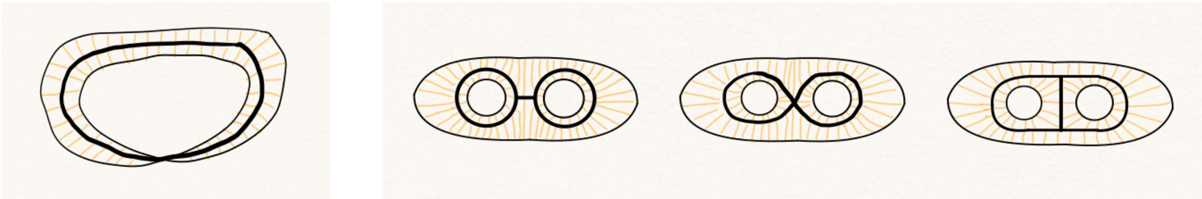

Figure 2: Deformation retracts (bold subsets) of sets (yellow), with thin lines indicating retraction map. Left: a Möbius band retracting onto a circle. Right: a disk with two smaller sub-disks removed deformation retracts to three different spaces.

6.2 Homotopy equivalence

Definition 6.4 (Homotopy equivalence)**.** Two metric spaces X and Y are *homotopy equivalent* if there exist continuous maps

$$f : X \to Y, \qquad g : Y \to X$$

such that

$$g \circ f \simeq \mathrm{id}_X \qquad \text{and} \qquad f \circ g \simeq \mathrm{id}_Y.$$

In this case we write $X \simeq_h Y$ and say that X and Y have the same *homotopy type.*

Proposition 6.5. *If A is a deformation retract of X, then X and A are homotopy equivalent.*

Idea. Let H be a deformation retraction from X onto A, with $H_0 = \mathrm{id}_X$ and $H_1(X) = A$. Let $r := H_1 : X \to A$ (retraction) and $i : A \hookrightarrow X$ the inclusion. Then $r \circ i = \mathrm{id}_A$, and $i \circ r$ is homotopic to id_X via the homotopy H. Hence X and A are homotopy equivalent with $f = r$ and $g = i$. □

Example 6.6 (Contractible sets)**.** The ball $\overline{B}_1(0)$ deformation retracts onto $\{0\}$. Hence $\overline{B}_1(0) \simeq_h \{0\}$: they have the same homotopy type.

Example 6.7 (Annulus and circle)**.** Consider the annulus

$$A = \{x \in \mathbb{R}^2 : 1 \leq \|x\| \leq 2\}.$$

Using radial projection, one can construct a deformation retraction $A \to S^1$; therefore $A \simeq_h S^1$. Although the annulus and the circle are not homeomorphic (one is 2-dimensional, the other is 1-dimensional as manifolds), they share the same "loop structure".

Definition 6.8 (Contractible space)**.** A metric space X is *contractible* if it is homotopy equivalent to a point.

Example 6.9. Any convex subset of $\mathbb{R}^d$ is contractible: fix a point x_0 in the set and use the straight-line homotopy $H(x,t) = (1-t)x + tx_0$ to contract the whole set to x_0.

For completeness, let us note that there exist sets that are contractible, but in which no point is a deformation retract of the whole space. The canonical example of such a set is called the *dunce hat* (as discussed [Hatcher]).

7 Exercises

Basic exercises

Exercise 7.1 (Homeomorphisms and compactness)**.** A subset $K \subset \mathbb{R}^d$ is said to be *compact* if every sequence in K has a convergent subsequence with limit in K.

(a) Show that the closed interval $[0,1]$ is compact.

(b) Show that the open interval $(0,1)$ is not compact.

(c) Let $f : [0,1] \to (0,1)$ be a continuous bijection. Show that f^{-1} cannot be continuous. (Hint: use the compactness of $[0,1]$ and the fact that a continuous image of a compact set is compact.) Conclude that $[0,1]$ and $(0,1)$ are not homeomorphic.

Exercise 7.2 (Path-connectedness)**.** Let $X \subset \mathbb{R}^2$ be the union of two disjoint closed disks.

(a) Prove that X is not path-connected.

(b) How many connected components does X have?

(c) Show that X is not homeomorphic to a single closed disk.

Further exercises

Exercise 7.3 (Explicit homotopy for punctured space)**.**

(a) Construct an explicit deformation retraction of $\mathbb{R}^d \setminus \{0\}$ onto S^{d-1}.

(b) Let $X = T^2 \setminus \{c\}$ be the 2-dimensional torus with one point deleted. What simpler space is it homotopy equivalent to?

Exercise 7.4 (Topology of basic spaces)**.** For each of the following spaces, give an *intuitive* description of its connected components and of how many distinct "loops" it has (in the everyday sense: closed curves that cannot be shrunk to a point without leaving the space).

(a) the circle $S^1 \subset \mathbb{R}^2$,

(b) the disk B^2,

(c) the flat cylinder $S^1 \times [0,1] \subset \mathbb{R}^3$,

(d) the 2-sphere $S^2 \subset \mathbb{R}^3$,

(e) the (surfacic) torus T^2,

(f) the skeleton of a 3D cube (i.e. the edges and corners of $[0,1]^3$).

Solutions

Exercise 1 (Homeomorphisms and compactness).

(a) Let $(x_n)_{n\geq 1}$ be a sequence in $[0,1]$. We will construct a subsequence of it converging in $[0,1]$.

- A first method consists in reasoning by dichotomy. Define $I_0 = [0,1]$ and, inductively, split I_k into two closed halves and choose $I_{k+1} \subset I_k$ to be a half that contains infinitely many terms of (x_n). Then $(I_k)_{k\geq 0}$ is a nested sequence of closed intervals with diameters $\mathrm{diam}(I_k) = 2^{-k} \to 0$, so $\bigcap_k I_k = \{x\}$ for some $x \in [0,1]$ (by completeness of $\mathbb{R}$). For each k, pick n_k such that $n_{k+1} > n_k$ and $x_{n_k} \in I_k$. Since $x_{n_k}, x \in I_k$ and $\mathrm{diam}(I_k) \to 0$, we have $x_{n_k} \to x$. Thus every sequence in $[0,1]$ has a convergent subsequence with limit in $[0,1]$, so $[0,1]$ is compact.
- Another method consists in considering $\bar{x} := \limsup_{n\to\infty} x_n = \lim_{n\to\infty} \big(\sup_{k\geq n} u_k\big)$, which is, by construction, a converging subsequential limit of $(x_n)_n$. Indeed, there always exists a subsequence converging to $\bar{x}$. Furthermore, by the least-upper/lower-bound property in $\mathbb{R}$, this limit lies in $[0,1]$.

(b) Consider the sequence $x_n = \frac{1}{n}$ in $(0,1)$. We know $x_n \to 0$ in $\mathbb{R}$, but $0 \notin (0,1)$. Any subsequence of (x_n) still converges to 0, so no subsequence has its limit in $(0,1)$. Thus $(0,1)$ fails the definition of compactness.

(c) Suppose, for contradiction, that $f : [0,1] \to (0,1)$ is a continuous bijection and that f^{-1} is continuous. Then f is a homeomorphism between $[0,1]$ and $(0,1)$.

Since $[0,1]$ is compact and f is continuous, the image $f([0,1])$ is compact. But f is surjective onto $(0,1)$, so $(0,1)$ would be compact. This contradicts point (b).

Hence f^{-1} cannot be continuous, and therefore $[0,1]$ and $(0,1)$ are not homeomorphic.

Exercise 2 (Path-connectedness). Let $X \subset \mathbb{R}^2$ be the union of two disjoint closed disks.

(a) Call the two disks D_1 and D_2. They are disjoint and there is a positive distance between them. Suppose there is a path $\gamma : [0,1] \to X$ with $\gamma(0) \in D_1$ and $\gamma(1) \in D_2$. Since γ is continuous and the closed disks are closed sets, the sets

$$A = \{t \in [0,1] : \gamma(t) \in D_1\}, \quad B = \{t \in [0,1] : \gamma(t) \in D_2\}$$

are closed in $[0,1]$, A is non-empty (contains 0), and B is non-empty (contains 1). Because D_1 and D_2 are disjoint and $X = D_1 \cup D_2$, every point of $\gamma([0,1])$ lies in exactly one of D_1 or D_2, so $A \cup B = [0,1]$ and $A \cap B = \emptyset$.

Thus $[0,1]$ is the disjoint union of two non-empty closed sets A and B, contradicting the fact that $[0,1]$ is connected. Hence no such path can exist, and X is not path-connected.

(b) By definition, $X = D_1 \cup D_2$ with D_1, D_2 disjoint, and each disk is path-connected (and hence connected). There is no path from a point in D_1 to a point in D_2. Therefore each disk forms its own connected component, so X has exactly two connected components.

(c) A closed disk is connected and has exactly one connected component. The space X has two connected components. Since the number of connected components is a topological invariant (it is preserved under homeomorphisms), X cannot be homeomorphic to a single closed disk.

Exercise 3 (Explicit homotopy of punctured space).

(a) Define $H : (\mathbb{R}^d \setminus \{0\}) \times [0,1] \to \mathbb{R}^d \setminus \{0\}$ by

$$H(x,t) = (1-t)x + tx/\|x\|.$$

For each fixed t, $H(\cdot,t)$ is continuous, and for each fixed x, $H(x,\cdot)$ is continuous, so H is continuous. We have $H(x,0) = x$ for all $x \in \mathbb{R}^d \setminus \{0\}$ (so $H_0 = \mathrm{id}_{\mathbb{R}^d\setminus\{0\}}$), $H(x,t) = x$ for all $x \in S^{d-1}$, $t \in [0,1]$, and $H(x,1) = x/\|x\| \in S^{d-1}$ for all $x \in \mathbb{R}^d \setminus \{0\}$.

Thus H is a deformation retract from $\mathbb{R}^d \setminus \{0\}$ onto S^{d-1}.

(b) In a nutshell, X is homotopy equivalent to two circles connected by one point. Seeing the torus as a unit square with opposite edges glued together, the homotopy equivalence can be obtained visually by replacing the point by a larger and larger square until it touches the edges (see Figure 3).

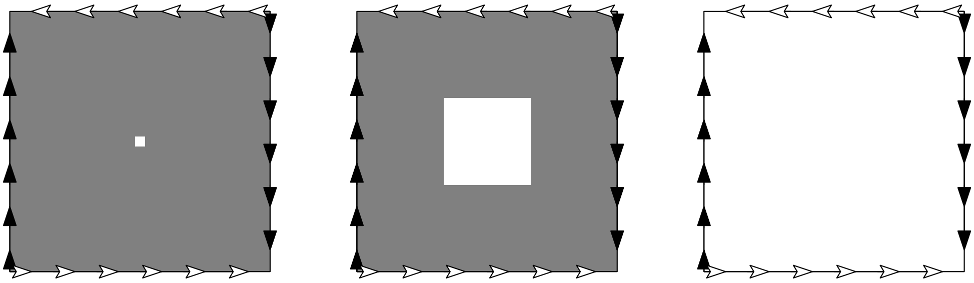

Figure 3: Homotopy of the punctured torus with the join of two circles.

If we want to be more formal, we can describe an explicit deformation retraction of the punctured torus onto the union of its meridian (horizontal) and a longitude (vertical) circle, intersecting in one point.

Step 1: Model the torus as a quotient of the square.

Write the torus as the quotient

$$T^2 \;=\; [0,1] \times [0,1]/\sim,$$

where we identify opposite edges:

$$(0,y) \sim (1,y), \qquad (x,0) \sim (x,1).$$

Under this quotient, all four corners are identified to a single point, which we take as the intersection point of the meridian and longitude circles. The vertical edge $\{0\} \times [0,1]$ becomes a meridian circle, and the horizontal edge $[0,1] \times \{0\}$ becomes a longitude circle. Their union in the quotient is the desired "figure-eight" graph:

$$G \;\subset\; T^2.$$

Choose a point in the *interior* of the square, say

$$c := \Big(\frac{1}{2}, \frac{1}{2}\Big),$$

and let $p \in T^2$ be its image under the quotient map

$$q : [0,1]^2 \to T^2.$$

We will construct a deformation retraction of $T^2 \setminus \{p\}$ onto G.

Step 2: Deformation retract the punctured square onto its boundary.

Consider the punctured square

$$S := [0,1]^2 \setminus \{c\}.$$

For each $x \in S$, $x \neq c$, consider the ray starting at c and passing through x:

$$\{c + \lambda(x - c) : \lambda \geq 0\}.$$

This ray meets the boundary $\partial[0,1]^2$ in a unique point, which we denote by $P(x)$. Geometrically, $P(x)$ is obtained by going from c through x until you first hit the boundary of the square.

Define

$$H_S : S \times [0,1] \to S, \qquad H_S(x,t) = (1-t)x + t\,P(x).$$

Then:

- $H_S(x,0) = x$ for all $x \in S$;
- $H_S(x,1) = P(x) \in \partial[0,1]^2$ for all $x \in S$;
- if $x \in \partial[0,1]^2$, then $P(x) = x$, hence $H_S(x,t) = x$ for all t.

Moreover, the segment between x and $P(x)$ never passes through c (since x lies between c and $P(x)$ on the ray), so H_S indeed has values in S for all t.

Thus H_S is a deformation retraction of S onto the boundary $\partial[0,1]^2$.

Step 3: Pass to the quotient torus and the figure-eight graph.

The quotient map $q : [0,1]^2 \to T^2$ identifies opposite edges but fixes each boundary point up to this identification. The homotopy H_S keeps the boundary *pointwise fixed*, so it is compatible with the quotient: for any (x,t) and any $x' \sim x$, we have

$$q\big(H_S(x,t)\big) = q\big(H_S(x',t)\big).$$

Therefore H_S induces a continuous map

$$H : (T^2 \setminus \{p\}) \times [0,1] \to T^2 \setminus \{p\}$$

such that

$$H\big(q(x),t\big) = q\big(H_S(x,t)\big).$$

By construction:

- $H(\cdot,0) = \mathrm{id}_{T^2\setminus\{p\}}$,
- $H(\cdot,1)$ maps every point to the image of the boundary, which is exactly the 1-skeleton of the cell decomposition of T^2, i.e. the union of the meridian and longitude circles intersecting at the corner point,
- points of this graph (the image of $\partial[0,1]^2$) are fixed by H for all t.

Thus H is a deformation retraction of $T^2 \setminus \{p\}$ onto the graph G consisting of the meridian and longitude circles intersecting at one point.

Exercise 4 (Topology of basic spaces). We only ask for intuitive descriptions.

(a) *Circle* S^1. It is connected and path-connected: you can move continuously from any point to any other along the circle. There is one "essential loop": going once around the circle. This loop cannot be shrunk to a point without leaving the circle.

(b) *Disk* B^2. It is connected and path-connected. Every loop in the disk can be shrunk continuously to a point while staying inside the disk (you can tighten it like a rubber band until it collapses). Intuitively, the disk has no 1-dimensional "hole".

(c) *Flat cylinder* $S^1 \times [0,1]$. Think of this as a tube or a can without top and bottom. It is connected and path-connected. There is one essential loop going around the cylinder (like the equator on a can). Such a loop cannot be shrunk to a point without cutting through the cylinder. Loops running vertically from bottom to top can be contracted to a point.

(d) *2-sphere* S^2. The ordinary sphere is connected and path-connected. Any closed curve on the sphere can be contracted to a point while staying on the sphere (you can slide it over the surface). So, in the intuitive sense of 1-dimensional loops, the 2-sphere has no nontrivial loop that goes around a "hole".

(e) *Torus* T^2. This is the surface of a doughnut. It is connected and path-connected. It has two different types of essential loops: (1) loops going around the hole (the "long way"), and (2) loops going around the tube (the "short way"). Neither kind can be shrunk to a point without leaving the surface; they represent two independent 1-dimensional "holes". Furthermore, the torus encloses one 2-dimensional void.

(f) *Skeleton of a cube.* Take the union of the 12 edges and 8 vertices of a cube. This space is connected and path-connected: you can move along the edges from any vertex to any other. It has 5 independent "loop". Indeed, when drawn as a planar graph, we see that the "exterior loop" is just enclosing the 5 others, so counting it would be redundant.